K26P 1215

Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree (CBCSS – OBE – Reg./Supple./Imp.) Examination, April 2026
(2023 Admission Onwards)
MATHEMATICS/MATHEMATICS (Multivariate Calculus and Mathematical Analysis, Modelling and Simulation, Financial Risk Management)
MSMAT02C07/MSMAF02C07 : Measure Theory

Time : 3 Hours　　　　　　　　　　　　　　　　　　Max. Marks : 80

PART – A

Answer **any 5** questions from this Part. **Each** question carries 4 marks.

1. Define Lebesgue outer measure of a set and show that it is translation invariant.
2. Define essential supremum of a measurable function f and prove that $f \le$ ess sup f, a.e.
3. Consider the function f defined on [0, 1] by : f(x) = 0 for x rational and if x is irrational, f(x) = n, where n is the number of zeros immediately after the decimal point, in the representation of x on the decimal scale. Show that f is measurable and find $\int_0^1 f\,dx$.
4. Define a complete measure and a σ-finite measure on a ring $\mathcal{R}$ of sets. Show that the Lebesgue measure m defined on the class $\mathcal{M}$ of all Lebesgue measurable sets is σ-finite and complete.

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5. Show that $\lim_{n\to\infty}\int_0^1 f_n(x)dx = 0$ where $f_n(x) = \frac{\log(x+n)}{n}e^{-x}\cos x$.
6. Let $\{f_n\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty}\int|f_n|d\mu < \infty$.

   Then prove that $\sum_{n=1}^{\infty} f_n$ converges a.e., its sum, f, is integrable, and

   $\int f\,d\mu = \sum_{n=1}^{\infty}\int f_n\,d\mu$.　　　　**(5×4=20)**

PART – B

Answer **any 3** questions from this Part. **Each** question carries **7** marks.

7. Let c be any real number and let f and g be real-valued measurable functions defined on the same measurable set E. Then show that f + c, cf, f + g, f – g and fg are also measurable. Further, if h is continuous on E, then prove that h is measurable.
8. Prove that there exists a set which is not Lebesgue measurable.
9. Let f and g be non-negative measurable functions. Then show that $\int f\,dx + \int g\,dx = \int (f+g)dx$. Also prove the monotonicity of the integrals in this case.
10. Let $\{E_i\}$ be a sequence of measurable sets such that $E_1 \supseteq E_2 \supseteq ....$ and $\mu(E_1) < \infty$. Then prove that $\mu(\lim E_i) = \lim \mu(E_i)$. Does the result hold if the condition that $\mu(E_1) < \infty$ is dropped ? Justify.
11. State and prove Hölder's Inequality for a general measure space X.　　　　**(3×7=21)**

PART – C

Answer **any 3** questions from this Part. **Each** question carries **13** marks.

12. Prove that the Lebesgue outer measure of an interval equals its length. Further prove that Lebesgue outer measure of a countable set is zero.
13. Suppose that f is any extended real-valued function which for every x and y satisfies f(x) + f(y) = f(x + y).
    i) Show that f is either everywhere finite or everywhere infinite.
    ii) Show that if f is measurable and finite, then f(x) = xf (1) for each x.

14. State and prove Fatou's Lemma and the Monotone Convergence Theorem for Lebesgue measurable functions. Give an example to show that we may have strict inequality in Fatou's Lemma.
15. If μ is a measure on a ring $\mathcal{R}$ and if the set function μ^* is defined on $\mathcal{H}(\mathcal{R})$ by

    $$\mu^*(E) = \inf\left[\sum_{n=1}^{\infty}\mu(E_n) : E_n \in \mathcal{R},\ n = 1, 2, ..., E \subseteq \bigcup_{n=1}^{\infty} E_n\right],$$

    then prove that for $E \in \mathcal{R}$, $\mu^*(E) = \mu(E)$. Further prove that μ^* is an outer measure on $\mathcal{H}(\mathcal{R})$.
16. a) Let f be a measurable function, $f : X \to [0, \infty]$. Then prove that there exists a sequence $\{\phi_n\}$ of measurable simple functions such that, for each x, $\phi_n(x) \uparrow f(x)$.

    b) Let $\{f_n\}$ be a sequence of measurable functions, $f_n : X \to [0, \infty]$; then show that $\int \sum_{n=1}^{\infty} f_n\,d\mu = \sum_{n=1}^{\infty}\int f_n\,d\mu$.　　　　**(3×13=39)**