K26P 0722

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (C.B.C.S.S. – O.B.E. – Regular/Supple./Imp.) Examination, April 2026

(2023 Admission Onwards)

MATHEMATICS

MSMAT04C14 : Operator Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any five** questions. **Each** question carries 4 marks. **(5×4=20)**

1. Show that a sequence (x_n) in a Banach space is bounded if and only if $(f(x_n))$ is bounded for every $f \in X^*$, using Uniform Boundedness Principle.

2. Define a closed linear map. State open mapping theorem and closed graph theorem.

3. Let X be a complex Banach spaces, $T : X \to X$ a linear map, and $\lambda \in \rho(T)$. Show that $R_\lambda(T)$ is defined on the whole space X, and is bounded, if T is closed or bounded.

4. Give example for a linear map T such that the spectrum $\sigma(T)$ of T is not compact.

5. Give example for a bounded linear map which is not compact. Give all the details.

6. Define a positive linear map. Show that if $T \geq 0$, then $\sigma(T)$ will contain no negative real numbers.

PART – B

Answer **any three** questions. **Each** question carries 7 marks. **(3×7=21)**

7. Give example for a normed space, which cannot become a Banach space under any norm. Give all the details.

8. a) Show that the collection CL(X) of all compact operators on a Banach space X is closed with respect to the operator norm.

   b) Prove that the operator T defined on l^2 space by $T(\xi_1, \xi_2, \ldots) = (\xi_1, \frac{\xi_2}{2}, \ldots)$ is compact.

P.T.O.

9. a) Show that the collection CL(X) of all compact operators on a Banach space X is a two-sided ideal in B(X), the space of all bounded operators on X.

   b) If $T : H \to H$ is a bounded operator, show that T is compact if and only if T^*T is compact.

10. Show that the residual spectrum $\sigma_r(T)$ of a bounded self-adjoint operator on a Hilbert space H is empty.

11. a) Define projections on a Hilbert space and show that a bounded operator $P : H \to H$ on a Hilbert space H is a projection if and only if $P = P^2 = P^*$.

    b) Show that if P is a projection on a Hilbert space H, then $0 \leq P \leq I$ and $\|P\| = 1$, if P is not the zero operator.

PART – C

Answer **any three** questions. **Each** question carries 13 marks. **(3×13=39)**

12. a) Define reflexive space and show that all finite-dimensional normed spaces are reflexive.

    b) State and prove Uniform Boundedness Theorem.

13. a) If $X \neq \{0\}$ is a complex Banach space and T is a bounded linear operator on X, then show that the spectrum of T is non-empty.

    b) If $X \neq \{0\}$ is a complex Banach space and T is a bounded linear operator on X, then show that the spectral radius $r_\sigma(T)$ of T satisfies, $r_\sigma(T) = \lim_{n \to \infty} \|T^n\|^{1/n}$.

14. Let $T : X \to X$ be a compact linear operator on a normed space X. Then show that for every $\lambda \neq 0$, the range of $T - \lambda I$ is closed.

15. a) Show that the spectrum of a bounded self-adjoint operator on a complex Hilbert space H is real.

    b) Show that the spectrum of a bounded self-adjoint operator on a complex Hilbert space H is a subset of [m, M], where $m = \inf_{x \in H, \|x\|=1} \langle Tx, x \rangle$ and $M = \sup_{x \in H, \|x\|=1} \langle Tx, x \rangle$. Also show that m, M are spectral values.

16. a) If P_1, P_2 are projections on Y_1, Y_2 respectively, then show that $P_2 - P_1$ is a projection if and only if $Y_1 \subseteq Y_2$.

    b) Show that if $P_2 - P_1$ is a projection, then its range is $Y_1^\perp \cap Y_2$.