K25P 3666

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (C.B.S.S. – Supplementary) Examination, October 2025 (2022 Admission) MATHEMATICS MAT1C03 : Real Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. When can you say that a set is convex ? Prove that balls are convex.

2. Prove that a set E is open if and only if its complement is closed.

3. i) When can you say that a function f is said to be differentiable at a point x ?
   ii) Suppose f and g are defined on [a, b] and are differentiable at a point $x \in [a, b]$. then prove that fg is differentiable at x and $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

4. Suppose f is differentiable in (a, b). If $f'(x) \le 0$ for all $x \in (a, b)$, then prove that f is monotonically decreasing.

5. If f is monotonic on [a, b], then prove that the set of discontinuities of f is countable.

6. Give an example to show that boundedness of f' is not necessary for f to be of bounded variation. Justify.

P.T.O.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

(4×16=64)

UNIT – I

7. a) Suppose $K \subset Y \subset X$. Then prove that K is compact relative to X if and only if K is compact relative to Y.

   b) Prove that compact subsets of metric spaces are closed.

8. a) If a set E in R^k has one of the following three properties, then prove that it has the other two :

   i) E is closed and bounded

   ii) E is compact

   iii) Every infinite subset of E has a limit point in E

   b) If $\{I_n\}$ is a sequence of intervals in R^1, such that $I_n \supset I_{n+1}$, (n = 1, 2, 3, ...), then prove that $\bigcap_{n=1}^{\infty} I_n$ is not empty.

9. Let E be a noncompact set in R^1. Then prove that,

   a) There exists a continuous function on E which is not bounded.

   b) If in addition, E is bounded, then prove that there exists a continuous function on E which is not uniformly continuous.

   c) There exists a continuous function on E which is not uniformly continuous.

UNIT – II

10. a) Suppose f is a real differentiable function on [a, b] and suppose $f'(a) < \lambda < f'(b)$. Then prove that there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

    b) State and prove L'Hospital's rule.

    c) Give an example to show that the L'Hospital's rule fails to be true for complex-valued functions. Justify.

11. a) Show that $f \in R(\alpha)$ on [a, b] if and only if for every $\epsilon > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

    b) Prove that if f is continuous on [a, b], then $f \in R(\alpha)$ on [a, b].

    c) If f is monotonic on [a, b] and if α is continuous on [a, b] then prove that $f \in R(\alpha)$.

12. a) Assume α increases monotonically and $\alpha' \in R$ on [a, b]. Let f be a bounded real function on [a, b]. Then prove that $f \in R(\alpha)$ if and only if $f\alpha' \in R$ and $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.

b) Suppose ϕ is a strictly increasing continuous function that maps an interval [A, B] onto [a, b]. Suppose α is monotonically increasing on [a, b] and $f \in R(\alpha)$ on [a, b]. Define β and g on [A, B] by $\beta(y) = \alpha(\phi(y))$, $g(y) = f(\phi(y))$. Then prove that $g \in R(\beta)$ and $\int_a^b g d\beta = \int_a^b f d\alpha$.

UNIT – III

13. a) State and prove the fundamental theorem of calculus.

b) State and prove the formula for "integration by parts".

c) If f and F map [a, b] into R^k, if $f \in R$ on [a, b] and if $F' = f$, then prove that $\int_a^b f(t)dt = F(b) - F(a)$.

14. a) Define Total Variation of f on the interval [a, b]. Prove that if f and g are each of bounded variation on [a, b], then so are their sum, difference and product. Also prove that $V_{f\pm g} = V_f + V_g$ and $V_{f.g} = AV_f + BV_g$, where $A = \sup\{|g(x)| : x \in [a, b]\}$, $B = \sup\{|f(x)| x \in [a, b]\}$.

b) Let f be of bounded variation on [a, b] and assume that f is bounded away from zero; that is, suppose that there exists a positive number m such that $0 < m \leq |f(x)|$ for all x in [a, b]. Then prove that $g = \frac{1}{f}$ is also of bounded variation on [a, b] and $V_g \leq V_{\frac{f}{m^2}}$.

15. a) Let f be defined on [a, b]. Then prove that f is of bounded variation on [a, b] if and only if f can be expressed as the difference of two increasing functions.

b) Let f be of bounded variation on [a, b]. If $x \in (a, b]$, let $V(x) = V_f(a, x)$ and $V(a) = 0$. Then prove that every point of continuity of f is also a point of discontinuity of V. Also prove that the converse is also true.