K25P 1136

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (C.B.S.S. – Supple./Imp.) Examination, April 2025 (2021 and 2022 Admissions) MATHEMATICS MAT4C16 : Differential Geometry

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks : **(4×4=16)**

1. Show that the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F : \mathbb{R}^{n+1} \to \mathbb{R}$.
2. Define gradient vector field. Find the gradient vector field of the function $f(x_1, x_2) = x_1^2 + 2x_2^2$, $x_1, x_2 \in \mathbb{R}$.
3. Show that SL(2) is a 3-surface in $\mathbb{R}^4$.
4. Define differential 1-form. Give an example.
5. Find the length of the parametrized curve $\alpha : [0, 2] \to \mathbb{R}^2$ defined by $\alpha(t) = (t^2, t^3)$.
6. Obtain the torus as a parametrized surface in $\mathbb{R}^3$.

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **16** marks : **(4×16=64)**

Unit – I

7. a) Prove that a connected n-surface has exactly 2 orientations.

   b) Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and let c = f(p). Then prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

P.T.O.

8. a) State and prove Lagrange's multiplier theorem.

b) Show that the surface of revolution is a 2-surface.

c) Show that the maximum and minimum of the function $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$, $a, b, c \in \mathbb{R}$ on the unit circle $x_1^2 + x_2^2 = 1$ are the eigenvalues of the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

9. a) Let $\mathbb{X}$ be the vector field defined by $\mathbb{X}(x_1, x_2) = (x_1, x_2, -x_2, x_1)$. Find the integral curve of $\mathbb{X}$ passing through (1, 0).

b) Let S be an n–surface in $\mathbb{R}^{n-1}$, let $\mathbb{X}$ be a smooth tangent vector field on S and let $p \in S$. Then prove that there exist an open interval I containing 0 and a parametrized curve $\alpha : I \to S$ such that (i) $\alpha(0) = p$, (ii) $\dot{\alpha}(t) = \mathbb{X}(\alpha(t))$, (iii) If $\beta : \tilde{I} \to U$ is any other integral curve of $\mathbb{X}$ with $\beta(0) = p$ then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)\ \forall t \in \tilde{I}$.

Unit – II

10. a) Sketch the spherical image of the hyperbola $x_1^2 - x_2^2 = 4, x_1 > 0$.

b) Show that for each pair of orthogonal unit vectors $\{e_1, e_2\}$ in $\mathbb{R}^3$, $\alpha(t) = \cos ate_1 + \sin ate_2$ is a geodesic on the unit sphere.

c) Compute $\nabla_v f$, given that $f(x_1, x_2) = 2x_1^2 + 3x_2^2$ at $v = (1, 0, 2, 1)$.

11. a) Let S be an n-surface in $\mathbb{R}^3$, let $p, q \in S$ and let α be a piecewise smooth parametrized curve from p to q. Prove that the parallel transport $P_\alpha : S_p \to S_q$ along α is a vector space isomorphism which preserves dot product.

b) Let S be a 2–surface in $\mathbb{R}^3$ and let $\alpha : I \to S$ be a geodesic on S with $\dot{\alpha} \neq 0$. Then prove that a vector field $\mathbb{X}$ tangent to S along α is parallel along α if and only if both norm of $\mathbb{X}$ and the angle between $\mathbb{X}$ and $\dot{\alpha}$ are constants along α.

12. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n-1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in S$. Then prove that the Gauss map maps S onto the unit sphere S^n.

Unit – III

13. a) Let C be a connected oriented plane curve and let $\beta : I \to C$ be a unit speed global parametrization of C. Then prove that β is either one to one or periodic. Also prove that β is periodic if and only if C is compact.

    b) Let $\alpha : I \to \mathbb{R}^{n-1}$ be a parametrized curve and if $\beta : I \to \mathbb{R}^{n-1}$ is a reparametrization of α then prove that $l(\alpha) = l(\beta)$.

14. a) Find the principal curvatures at p and the principal curvature directions at p of the hyperboloid $-x_1^2 - x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$ at p = (0, 0, 1).

    b) Find the Gaussian curvature of the ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$, a, b, c all $\neq 0$ oriented by its outward normal.

15. a) State and prove the inverse function theorem for n-surfaces.

    b) Let S be an n-surface in $\mathbb{R}^{n-1}$ and let $f : S \to \mathbb{R}^k$. Then prove that f is smooth if and only if $f \circ \varphi : U \to \mathbb{R}^k$ is smooth for each local parametrization $\varphi : U \to S$.
