



K25P 1135

Reg. No. :

Name :

**IV Semester M.Sc. Degree (C.B.S.S. – Supple./Imp.) Examination, April 2025
(2021 and 2022 Admissions)**

**MATHEMATICS
MAT4C15 : Operator Theory**

Time : 3 Hours

Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Let X be a normed space and $A \in BL(X)$. Then prove that A is invertible if and only if A is bounded below and surjective.

2. Let X and Y be normed spaces. Let F_1 and F_2 be in $BL(X, Y)$ and $k \in K$. Then prove that

$$(F_1 + F_2)' = F_1' + F_2' \text{ and } (kF_1)' = kF_1'.$$

3. If $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$ in a normed space X , then show that $x_n + y_n \xrightarrow{w} x + y$.

4. Let X and Y be normed spaces and $F : X \rightarrow Y$ be linear. Prove that F is a compact map if and only if for every bounded sequence (x_n) in X , $(F(x_n))$ has a subsequence which converges in Y .

5. Let H be a Hilbert space and $A \in BL(H)$. Then prove that $Z(A) = R(A^*)^\perp$ and $Z(A^*) = R(A)^\perp$.

6. Define numerical range of an operator on a Hilbert space and prove or disprove that it is a closed subset of K .

P.T.O.



PART – B

Answer **any four** questions from this Part without omitting **any** Unit. Each question carries **16** marks. (4×16=64)

Unit – I

7. a) Let X be a normed space and $A \in BL(X)$ be of finite rank. Then prove that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.

- b) Let X be a Banach space over K and $A \in BL(X)$. Let $k \in K$ such that $|k|^p > \|A^p\|$, for some positive integer p . Then prove that $k \notin \sigma(A)$ and

$$(A - kI)^{-1} = \sum_{n=0}^{\infty} \frac{A^n}{k^{n+1}}.$$

8. a) Let X be a nonzero Banach space over C and $A \in BL(X)$. Then prove that

$$r_\sigma(A) = \inf_{n=1,2,\dots} \|A^n\|^{1/n} = \lim_{n \rightarrow \infty} \|A^n\|^{1/n}.$$

- b) Let X be a normed space and X_0 be a dense subspace of X . For $x' \in X'$, let $F(x')$ denote the restriction of x' to X_0 . Then prove that the map F is a linear isometry from X' onto X'_0 .

9. a) Let X be a normed space and (x_n) be a sequence in X . Then prove that (x_n) is weak convergent in X if and only if (i) (x_n) is a bounded sequence in X and (ii) there is some $x \in X$ such that $x'(x_n) \rightarrow x'(x)$ for every x' in some subset of X' whose span is dense in X' .

- b) Let X be a separable normed space. Then prove that every bounded sequence in X' has a weak* convergent subsequence.

Unit – II

10. Let X be a normed space. Then prove that X is reflexive if and only if every bounded sequence in X has a weak convergent subsequence.

11. a) Let X and Y be normed spaces and $F : X \rightarrow Y$ be linear. If F is continuous and of finite rank, then prove that F is a compact map and $R(F)$ is closed in Y . Conversely, prove that if X and Y are Banach spaces, F is a compact map and $R(F)$ is closed in Y , then F is continuous and of finite rank.

- b) Let X and Y be normed spaces and $F : X \rightarrow Y$ be linear. Let X be reflexive and $F(x_n) \rightarrow F(x)$ in Y whenever $x_n \xrightarrow{w} x$ in X . Then prove that $F \in CL(X, Y)$.



12. a) Let X be a normed space. $A \in CL(X)$ and $0 \neq k \in K$. If (x_n) is a bounded sequence in X such that $A(x_n) - kx_n \rightarrow y$ in X , then prove that there is a subsequence (x_{n_j}) of (x_n) such that $x_{n_j} \rightarrow x$ in X and $A(x) - kx = y$.
- b) Let X be a normed space and $A : X \rightarrow X$. Let $0 \neq k \in K$ and Y be a proper closed subspace of X such that $(A - kI)(X) \subset Y$. Then prove that there is some $x \in X$ such that $\|x\| = 1$ and for all $y \in Y$,

$$\|A(x) - A(y)\| \geq \frac{k}{2}.$$

Unit – III

13. a) Let H be a Hilbert space and $A \in BL(H)$. Then prove that there is a unique $B \in BL(H)$ such that for all $x, y \in H$,
- $$\langle A(x), y \rangle = \langle x, B(y) \rangle.$$
- b) Let H be a Hilbert space and $A \in BL(H)$. Prove the following :
- $R(A) = H$ if and only if A^* is bounded below.
 - $R(A^*) = H$ if and only if A is bounded below.
14. a) Let H be a Hilbert space and $A \in BL(H)$. Let A be self-adjoint. Then prove that
- $$\|A\| = \sup\{|\langle A(x), x \rangle| : x \in H, \|x\| \leq 1\}$$
- and $A = 0$ if and only if $\langle A(x), x \rangle = 0$ for all $x \in H$.
- b) Let $A \in BL(H)$ be self-adjoint. Then prove that A or $-A$ is a positive operator if and only if $|\langle A(x), y \rangle|^2 \leq \langle A(x), x \rangle \langle A(y), y \rangle$ for all $x, y \in H$.
15. a) Let $H \neq \{0\}$ and $A \in BL(H)$ be self-adjoint. Then prove that
- $$\{m_A, M_A\} \subset \sigma_a(A) = \sigma(A) \subset [m_A, M_A].$$
- b) Let $H \neq \{0\}$. Let $A \in BL(H)$ be self-adjoint. Then prove that
- $$\|A\| = \max\{|m_A|, |M_A|\} = \sup\{|k| : k \in \sigma(A)\}.$$