



K25P 2017

Reg. No. :

Name :

II Semester M.Sc. Degree (C.B.S.S. – Supplementary)

Examination, April 2025

(2021 and 2022 Admissions)

MATHEMATICS

MAT 2C 07 : Measure and Integration

Time : 3 Hours

Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Prove that $m^*([x]) = 0$ for any $x \in \mathbb{R}$.
2. Prove that every Borel set is measurable.
3. Prove that if f and g are measurable. $|f| \leq |g|$ a.e., and g is integrable, then f is integrable.
4. Prove that the intersection of hereditary σ -rings is again a σ -ring.
5. Let $f = g$ a.e. (μ) , where μ is a complete measure. Show that if f is measurable, so is g .
6. If $\rho(f, g) = \|f - g\|_p$, then prove that, for $p \geq 1$, ρ is a metric on $L^p(\mu)$.

PART – B

Answer **any four** questions from this part without omitting **any Unit**.

Each question carries **16** marks.

(4×16=64)

Unit – I

7. a) Let M be the class of Lebesgue measurable sets. Prove that M is a σ -algebra.
b) Prove that Lebesgue outer measure is countably additive on disjoint measurable sets.
8. Prove that the following statements regarding the set E are equivalent
 - a) E is measurable
 - b) $\forall \varepsilon > 0$, there exists O , an open set, $O \supseteq E$ such that $m^*(O - E) \leq \varepsilon$
 - c) there exists G , a G_δ -set, $G \supseteq E$ such that $m^*(G - E) = 0$
 - d) $\forall \varepsilon > 0$, there exists F , a closed set, $F \subseteq E$ such that $m^*(E - F) \leq \varepsilon$
 - e) there exists F , a F_σ -set, $F \subseteq E$ such that $m^*(E - F) = 0$.

P.T.O.



9. a) Show that there exists uncountable sets of zero measure.
b) Prove that there exists a non-measurable set.

Unit – II

10. Define integrable functions. Let f and g be integrable functions. Then prove the following.
- af is integrable and $\int afdx = a \int fdx$
 - $f + g$ is integrable and $\int (f + g)dx = \int fdx + \int gdx$
 - If $f = 0$ a.e., then $\int fdx = 0$
 - If $f \leq g$ a.e., then $\int fdx \leq \int gdx$
 - If A and B are disjoint measurable sets, then $\int_A fdx + \int_B fdx = \int_{A \cup B} fdx$.
11. a) State and prove Lebesgue's Dominated Convergence Theorem.
b) Show that $\lim_{\beta \rightarrow \infty} \int_a^b f(x) \sin \beta x dx = 0$.
12. a) The outer measure μ^* on $H(R)$ defined by μ on R and the corresponding outer measure defined by $\bar{\mu}$ on $S(R)$ and $\bar{\mu}$ on S^* are the same.
b) If μ is a σ -finite measure on a ring R , then prove that it has a unique extension to the σ -ring $S(R)$.

Unit – III

13. a) Let $0 < p < 1$ and $f \geq 0, g \geq 0, f, g \in L^p(\mu)$. Show that $\|f + g\|_p \geq \|f\|_p + \|g\|_p$.
b) Let $\{f_n\}$ is a sequence in $L^\infty(\mu)$ such that $\|f_n - f_m\|_\infty \rightarrow 0$ as $n, m \rightarrow \infty$, then prove that there exist a function f such that $\lim f_n = f$ a.e., $f \in L^\infty(\mu)$ and $\lim \|f_n - f\|_\infty = 0$.
14. Let $p \geq 1$ and $f, g \in L^p(\mu)$, then prove that $\left(\int |f + g|^p d\mu \right)^{\frac{1}{p}} \leq \left(\int |f|^p d\mu \right)^{\frac{1}{p}} + \left(\int |g|^p d\mu \right)^{\frac{1}{p}}$.
When does the equality occur ? Justify your answer.
15. State and prove Holder's inequality. When does the equality occur ? Justify.
-