



K24P 0862

Reg. No. :

Name :

**Second Semester M.Sc. Degree (CBSS – Supple. (One Time Mercy
Chance)/Imp.) Examination, April 2024
(2017 to 2022 Admissions)
MATHEMATICS
MAT2C07 : Measure and Integration**

Time : 3 Hours

Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Define Lebesgue outer measure. Show that $m^*(A) \leq m^*(B)$ if $A \subseteq B$.
2. Prove that, for any set A and any $\varepsilon > 0$ there is an open set O containing A and such that $m^*(O) \leq m^*(A) + \varepsilon$.
3. Show that if f is integrable, then f is finite valued a.e.
4. Show that there exist a smallest ring and a smallest σ -ring containing a given class of subsets of a space.
5. Define measure space and measurable space. Give examples.
6. Prove that if $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^q(\mu) \subseteq L^p(\mu)$.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

(4×16=64)

Unit – I

7. a) Prove that the following statements regarding the set E are equivalent.
 - i) E is measurable
 - ii) $\forall \varepsilon > 0$, there exists O , an open set, $O \supseteq E$ such that $m^*(O - E) \leq \varepsilon$

P.T.O.



- iii) there exists G , a G_δ -set, $G \supseteq E$ such that $m^*(G - E) = 0$
- iv) $\forall \varepsilon > 0$, there exists F , a closed set, $F \subseteq E$ such that $m^*(E - F) \leq \varepsilon$
- v) there exists F , a F_σ -set, $F \subseteq E$ such that $m^*(E - F) = 0$
- b) Show that every countable set has measure zero.
- 8. a) Show that the class M of Lebesgue measurable sets is a σ -algebra.
- b) Show that there exists uncountable sets of zero measure.
- 9. a) Prove that Lebesgue outer measure is countably additive on disjoint measurable sets.
- b) Prove that not every measurable set is a Borel set.

Unit – II

- 10. a) Let f be bounded and measurable on a finite interval $[a, b]$ and let $\varepsilon > 0$. Then show that there exist:
 - i) a step function h such that $\int_a^b |f - h| dx < \varepsilon$.
 - ii) a continuous function g such that g vanishes outside a finite interval and $\int_a^b |f - g| dx < \varepsilon$.
- b) Show that if $\alpha > 1$,

$$\int_0^1 \frac{\sin x}{1 + (nx)^\alpha} dx = O(n^{-1}) \text{ as } n \rightarrow \infty.$$
- 11. a) Show that $H(\mathbb{R}) = \{E : E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \mathcal{R}\}$.
- b) Let f be a bounded function defined on the finite interval $[a, b]$, then prove that f is Riemann integrable over $[a, b]$ if and only if it is continuous a.e.
- 12. a) Show that if μ is a σ -finite measure on $\mathbb{R}$, then the extension $\bar{\mu}$ of μ is also σ -finite.
- b) If μ is a σ -finite measure on a ring $\mathcal{R}$, then prove that it has a unique extension to the σ -ring $S(\mathcal{R})$.



Unit – III

13. If $1 \leq p < \infty$ and $\{f_n\}$ is a sequence in $L^p(\mu)$ such that $\|f_n - f_m\|_p \rightarrow 0$ as $n, m \rightarrow \infty$, then prove that there exist a function f and a subsequence $\{n_i\}$ such that $\lim f_{n_i} = f$ a.e. Also prove that $f \in L^p(\mu)$ and $\lim \|f_n - f\|_p = 0$.
14. a) State and prove Holder's Inequality. When does the equality occur ?
b) If $\rho(f, g) = \|f - g\|_p$, then prove that, for $p \geq 1$, ρ is a metric on $L^p(\mu)$.
15. Let $p \geq 1$ and $f, g \in L^p(\mu)$, then prove that

$$\left(\int |f + g|^p d\mu \right)^{\frac{1}{p}} \leq \left(\int |f|^p d\mu \right)^{\frac{1}{p}} + \left(\int |g|^p d\mu \right)^{\frac{1}{p}}$$

When does the equality occur ? Justify your answer.
