



K24P 4043

Reg. No. :

Name :

**I Semester M.Sc. Degree (CBSS – Supplementary)
Examination, October 2024
(2021 and 2022 Admissions)
MATHEMATICS
MAT1C04 : Basic Topology**

Time : 3 Hours

Max. Marks : 80

PART – A

Answer **four** questions from this part. **Each** question carries **4** marks.

1. Let (X, d) be a metric space, let $x \in X$ and let $\epsilon > 0$. Prove that $A = \{y \in X : d(x, y) \leq \epsilon\}$ is a closed subset of X .
2. Prove that every second countable space is separable. Is the converse true ? Justify your answer with an example.
3. Let $(A, \mathcal{T}_A)$ be a subspace of a topological space $(X, \mathcal{T})$. Prove that a subset C of A is closed in $(A, \mathcal{T}_A)$ if and only if there is a closed subset D of $(X, \mathcal{T})$ such that $C = A \cap D$.
4. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces, and let $(X_1 \times X_2, \mathcal{T})$ be the product space. Prove that the projection maps are continuous. Also show that the product topology is the smallest topology for which both projections are continuous.
5. A topological space $(X, \mathcal{T})$ is connected if and only if no nonempty proper subset of X is both open and closed.
6. Define Cantor set. **(4×4=16)**

P.T.O.



PART – B

Answer **four** questions from this part **without** omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let $\{\mathcal{T}_\alpha : \alpha \in \Lambda\}$ be a collection of topologies on a set X . Prove that $\bigcap \{\mathcal{T}_\alpha : \alpha \in \Lambda\}$ is a topology on X .
- b) Let X be a set and let $\mathcal{S}$ be a collection of subsets of X such that $X = \bigcup \{S : S \in \mathcal{S}\}$. Prove that there is a unique topology $\mathcal{T}$ on X such that $\mathcal{S}$ is a subbasis for $\mathcal{T}$.
- c) Let $X = \{1, 2, 3, 4, 5\}$ and $\mathcal{S} = \{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 5\}\}$. Prove that $\mathcal{S}$ is a subbasis for a topology on X . Also find $\mathcal{T}$.
8. a) Let A and B be subsets of a topological space $(X, \mathcal{T})$. Prove that :
- A is open if and only if $A = \text{int } A$.
 - $\text{int } (A) \subseteq \text{int } (B)$ whenever $A \subseteq B$.
 - $\text{int } (A \cap B) = \text{int } (A) \cap \text{int } (B)$.
 - $\text{int } (A) \cup \text{int } (B) \subseteq \text{int } (A \cup B)$.
- b) Let $n \in \mathbb{N}$ and $\mathcal{T}$ is the usual topology on $\mathbb{R}^n$. Prove that $(\mathbb{R}^n, \mathcal{T})$ is second countable.
9. a) Let $(X, \mathcal{T})$ be a topological space, Let $A \subset X$ and let $x \in X$. Prove that
- if there is a sequence of points of A that converges to x , then $x \in \bar{A}$.
 - if $(X, \mathcal{T})$ is first countable and $x \in \bar{A}$, then there is a sequence of points of A that converges to x .
- b) Let (X, d) be a complete metric space and let A be a subset of X with subspace metric $\rho = d|_{(A \times A)}$. Prove that (A, ρ) is complete if and only if A is a closed subset of X .
- c) Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $f : X \rightarrow Y$. Suppose $(X, \mathcal{T})$ is first countable and for each $x \in X$ and each sequence $\langle x_n \rangle$ such that $\langle x_n \rangle \rightarrow x$, the sequence $\langle f(x_n) \rangle \rightarrow f(x)$. Then prove that f is continuous.



Unit – II

10. a) Prove that the topological properties Hausdorff and metrizable are hereditary.
- b) Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be an indexed family of first countable spaces and let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Prove that $(X, \mathcal{T})$ is first countable if and only if $\mathcal{T}_\alpha$ is the trivial topology for all but a countable number of α .
11. a) Give an example to show that separability is not hereditary.
- b) State and prove Pasting lemma.
- c) Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces, and for $i = 1, 2$ let $\mathcal{B}_i$ be bases for $\mathcal{T}_i$. Then prove that $\mathcal{B} = \{U \times V : U \in \mathcal{B}_1 \text{ and } V \in \mathcal{B}_2\}$ is a basis for the product topology $\mathcal{T}$ on $X_1 \times X_2$.
12. a) Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be an indexed family of topological spaces, and for each $\alpha \in \Lambda$, let $(A_\alpha, \mathcal{T}_{A_\alpha})$ be a subspace of $(X_\alpha, \mathcal{T}_\alpha)$. Then prove that the product topology on $\prod_{\alpha \in \Lambda} A_\alpha$ is the same as the subspace topology on $\prod_{\alpha \in \Lambda} A_\alpha$ is determined by the product topology on $\prod_{\alpha \in \Lambda} X_\alpha$.
- b) Let $\{(Y_\alpha, \mathcal{U}_\alpha) : \alpha \in \Lambda\}$ be an indexed family of topological spaces. Let $\mathcal{U}$ be the product topology on $Y = \prod_{\alpha \in \Lambda} Y_\alpha$, let $(X, \mathcal{T})$ be a topological space, and let $f : X \rightarrow Y$ be a function. Prove that f is continuous if and only if $\pi_\alpha \circ f$ is continuous for each $\alpha \in \Lambda$.

Unit – III

13. a) Let $\mathcal{T}$ be the usual topology on $\mathbb{R}$. Prove that $(\mathbb{R}, \mathcal{T})$ is connected.
- b) State and prove intermediate value theorem.
- c) Prove that the Cantor set is totally disconnected.



14. a) Prove that the fixed point property is a topological invariant.
b) Prove that the topologist's sine curve is not pathwise connected.
15. a) Let $\{(A_\alpha, \mathcal{T}_{A_\alpha}) : \alpha \in \Lambda\}$ be a collection of connected subspaces of a topological space $(X, \mathcal{T})$ and let $A = \bigcup_{\alpha \in \Lambda} A_\alpha$. Then prove that
- i) If $\bigcap_{\alpha \in \Lambda} A_\alpha \neq \emptyset$ then $(A, \mathcal{T}_A)$ is connected.
 - ii) If $\Lambda = \mathbb{N}$ and $A_n \cap A_{n+1} \neq \emptyset$ for each $n \in \mathbb{N}$, then $(A, \mathcal{T}_A)$ is connected.
- b) Prove that a topological space $(X, \mathcal{T})$ is locally connected if and only if each component of each open set is open.
- c) Prove that every 0-dimensional T_0 space is totally disconnected. **(4×16=64)**
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