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K24U 0886

Reg. No. : ....

Name : ....

IV Semester B.Sc. Honours in Mathematics Degree (C.B.C.S.S. – Supplementary/One Time Mercy Chance) Examination, April 2024
(2016 to 2020 Admissions)
BHM 401 : ADVANCED REAL ANALYSIS AND METRIC SPACES

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark. **(4×1= 4)**

1. Define Cauchy sequence.
2. Define closed set.
3. Define norm of a partition.
4. Find $\lim \frac{nx}{1+x^2n^2}\ \forall\, x \in \mathbb{R}$.
5. State substitution theorem.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. Let $g(x) = \begin{cases} 2, & 0 \le x \le 1 \\ 3, & 1 < x \le 3 \end{cases}$, evaluate $\int_0^3 g(t)dt$.
7. Prove that every constant function on [a, b] is in $\mathcal{R}$[a, b].
8. If $f \in \mathcal{R}[a, b]$ and if $[c, d] \subseteq [a, b]$ then prove that the restriction of f to [c, d] is in $\mathcal{R}$[c, d].

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9. Check whether the sequence $(g_n) = \frac{x^n}{1+x^n}$ converge uniformly on [0, 2].
10. Let F_1, F_2 be subsets of a metric space (X, d). Prove that $\overline{F_1 \cup F_2} = \overline{F}_1 \cup \overline{F}_2$.
11. If A is a subset of the metric space (X, d), then prove that $d(A) = d(\overline{A})$.
12. Show that the function $f(x) = \frac{x}{1+x}, x \ge 0$ is monotonically increasing.
13. State and prove arithmetic mean-geometric mean inequality.
14. If Σa_n is an absolutely convergent series, then prove that the series $\Sigma a_n \cos nx$ is uniformly convergent.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks. **(8×4=32)**

15. State and prove fundamental theorem of Calculus (second form).
16. Let h(x) = x on [0, 1]. Using squeeze theorem show that h is Riemann integrable and evaluate the value of the integral.
17. If $f : [a, b] \to \mathbb{R}$ is continuous on [a, b], then prove that $f \in \mathcal{R}[a, b]$.
18. Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$.Then prove that f is continuous on A.
19. Let $h_n(x) = 2nxe^{-nx^2}$ for $x \in A = [0,1]$. Show that $\int_0^1 \lim h_n(x) \ne \lim \int_0^1 h_n(x)$.
20. Let X be any nonempty set with discrete metric d. Is (X, d) a complete metric space ? Justify.
21. Let X= C[0, 1], for $x, y \in X$ define $d(x, y) = \sup\{x(t) - y(t) : 0 \le t \le 1\}$.
Calculate the distance between x(t) = t and y(t)= t^2.

22. Let A be a subset of a metric space (X, d). Then prove that A° is an open subset of A that contains every open subset of A.
23. Let (X, d) be a metric space and F be a subset of X. Then prove that F is closed in X if and only if complement of F is open in X.
24. Let Y be a subspace of a metric space (X, d). Prove that every subset of Y that is open in Y is also open in X if and only if Y is open in X.
25. Prove that a mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y.
26. State and prove Cauchy-Hadamard theorem.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks. **(2×6=12)**

27. Show that $G_n(x) = x^n(1 - x)$ for $x \in A = [0,1]$ converges uniformly on A.
28. Let d be a metric on a nonempty set X. Show that the function e defined by $e(x, y) = \min\{1, d(x, y)\}$ where $x, y \in X$ is also a metric on X.
29. Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subseteq X$. Prove that a function $f : A \to Y$ is continuous at $a \in A$ if and only if whenever a sequence $\{x_n\}$ in A converges to a, the sequence $\{f(x_n)\}$ converges to f(a).
30. Show that Thomae's function is Riemann integrable on [0, 1].

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