K24U 0881

Reg. No. : ....

Name : ....

IV Semester B.Sc. Honours in Mathematics Degree (C.B.C.S.S. – O.B.E. – Regular/Supplementary/Improvement) Examination, April 2024
(2021 and 2022 Admissions)
4B14 BMH : ADVANCED REAL ANALYSIS

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark. **(4×1=4)**

1. Define bounded function.
2. If $I = [0, 4]$ calculate the norm of the partition $P = (0, 1, 1.5, 2, 3.4, 4)$.
3. Define uniform norm of a bounded function on a set.
4. C.P.V. $\int_{-\infty}^{\infty} x\,dx$ = ________________.
5. Define Beta function.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. Does the equation $f(x) = xe^x - 2$ has a root in $[0, 1]$? Justify your answer.
7. State Weierstrass Approximation Theorem.
8. Is every continuous function uniformly continuous ? Justify your answer.
9. Suppose that $f \in R[a, b]$ and $k \in \mathbb{R}$, then prove that $kf \in R[a, b]$.
10. State Substitution theorem. Can we apply the substitution theorem to evaluate the integral $\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\,dt$? Justify your answer.

P.T.O.

11. Let $g_n(x) = x^n$ for $x \in [0, 1]$. Does $(g_n(x))$ converge uniformly on $[0, 1]$? Justify your answer.
12. Define radius of convergence of a power series.
13. Check the convergence of the improper integral $\int_0^1 \frac{1}{x}\,dx$.
14. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks. **(8×4=32)**

15. State and prove Boundedness theorem.
16. Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then prove that f is uniformly continuous on I.
17. If $f : A \to \mathbb{R}$ is uniformly continuous on a subset A of $\mathbb{R}$ and if (x_n) is a Cauchy sequence in A, then prove that $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.
18. If $f \in R\,[a, b]$ then prove that f is bounded on $[a, b]$.
19. State and prove Fundamental Theorem of Calculus (Second Form).
20. Prove that $G(x) = \begin{cases} \frac{1}{n} & n \in \mathbb{N} \\ 0 & \text{elsewhere on } [0,1] \end{cases}$ is Riemann integrable on $[0, 1]$.
21. State Weierstrass M-Test and use it to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^3 + x^2 n^4}$ is uniformly convergent for all values of x.
22. Let $f(x) = \sum a_n x^n$ for $|x| < R$. If $f(x) = f(-x)$ for all $|x| < R$, show that $a_n = 0$ for all odd n.
23. If R is the radius of convergence of the power series $\sum a_n x^n$, then prove that the series is absolutely convergent if $|x| < R$ and is divergent if $|x| > R$, where $0 < R < \infty$.
24. Show that the improper integral $\int_{-\infty}^{\infty} \frac{1+x}{1+x^2}\,dx$ does not converge.

25. State and prove the relation between beta and gamma functions.
26. Express $\int_0^1 x^m (1 - x^n)^p\,dx$ in terms of Gamma functions and evaluate the integral $\int_0^1 x^5 (1 - x^3)^{10}\,dx$.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks. **(2×6=12)**

27. State and prove Location of Roots theorem.
28. Let $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. Then prove that $f \in R\,[a, b]$ if and only if its restrictions to $[a, c]$ and $[c, b]$ are both Riemann integrable and $\int_a^b f = \int_a^c f + \int_c^b f$.
29. State and prove Cauchy Criterion for Uniform Convergence (for sequence of functions).
30. Prove that the integral $\int_0^{\infty} \frac{1}{x^2 + \sqrt{x}}\,dx$ converges.
